THE ASTROPHYSICAL JOURNAL, 633:L13–L16, 2005 November 1
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EXACT VERSUS APPROXIMATE SOLUTIONS IN GAMMA-RAY BURST AFTERGLOWS

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Received 2005 July 18; accepted 2005 September 20; published 2005 October 12

ABSTRACT

We have recently obtained the exact analytic solutions for the relativistic equations relating the radial and time coordinates of a relativistic, thin uniform shell expanding in the interstellar medium in the fully radiative and fully adiabatic regimes. Here we reexamine the validity of the constant-index power-law relation between the Lorentz gamma factor and its radial coordinate, which is usually adopted in the gamma-ray burst (GRB) literature on the grounds of an "ultrarelativistic" approximation. Such expressions are found to be mathematically correct but only approximately valid in a very limited range of physical and astrophysical parameters and in an asymptotic regime that is reached for a very short time only, if at all, and they are shown to be nonapplicable to GRBs.

Subject headings: gamma rays: bursts — gamma rays: observations — ISM: kinematics and dynamics — relativity

1. INTRODUCTION

The discovery of afterglows (Costa et al. 1997) has us offered a very powerful tool for understanding gamma-ray bursts (GRBs). A consensus has been reached that such an afterglow originates from a relativistic, thin shell of baryonic matter propagating in the interstellar medium (ISM) and that its description can be obtained from the relativistic conservation laws of energy and momentum. In Bianco & Ruffini (2005) we reported the exact analytic solutions for the corresponding equations, respectively, under fully radiative and fully adiabatic conditions, giving, in both cases, explicit relations between the laboratory time and the radial coordinate of the shell. Here we compare and contrast our results with the simple constant-index power-law relation between the Lorentz gamma factor and the radial coordinate of the shell generally adopted in the literature and obtained using the so-called ultrarelativistic approximation (see, e.g., Sari 1997, 1998; Waxman 1997; Rees & Mészáros 1998; Granot et al. 1999; Panaitescu & Mészáros 1998, 1999; Chiang & Dermer 1999; Piran 1999; Gruzinov & Waxman 1999; van Paradijs et al. 2000; Mészáros 2002; and references therein). We show that such an approximation only holds in a very limited range of physical and astrophysical parameters and in an asymptotic regime that is reached for a very short time only, if at all. We demonstrate that this constant-index power law cannot be used for modeling GRBs. Illustrative examples are given for the source GRB 991216.

2. THE AFTERGLOW ANALYTIC SOLUTIONS

The fulfillment of the energy and momentum conservation for the equations of motion of the relativistic baryonic matter shell in the laboratory reference frame leads to the following equations (see, e.g., Piran 1999, Ruffini et al. 2003, and references therein):

$$dE_{\rm int} = (\gamma - 1)dM_{\rm ism}c^2, \tag{1a}$$

$$d\gamma = -[(\gamma^2 - 1)/M]dM_{\rm ism}, \tag{1b}$$

$$dM = [(1 - \epsilon)/c^2]dE_{\rm int} + dM_{\rm ism}, \tag{1c}$$

$$dM_{\rm ism} = 4\pi m_p n_{\rm ism} r^2\, dr, \tag{1d}$$

where M is the shell mass energy, $n_{\rm ism}$ is the ISM number density, m_p is the proton mass, ϵ is the emitted fraction of the energy developed in the collision with the ISM, and $M_{\rm ism}$ is the amount of ISM mass swept up by the shell within the radius r: $M_{\rm ism} = (4\pi/3)m_p n_{\rm ism}(r^3 - r_0^3)$, where r_0 is the starting radius of the baryonic matter shell. In general, an additional equation is needed in order to express the dependence of ϵ on the radial coordinate. In the following, ϵ is assumed to be constant, and such an approximation appears to be correct in the GRB context.

Consensus has also been reached on a simple integration of the equations of motion, equations (1a)–(1d), in the fully radiative case ($\epsilon = 1$; see Piran 1999, Ruffini et al. 2003, and Bianco & Ruffini 2005), leading to

$$\gamma = \frac{1 + (M_{\rm ism}/M_B)(1 + \gamma_0^{-1})[1 + (1/2)(M_{\rm ism}/M_B)]}{\gamma_0^{-1} + (M_{\rm ism}/M_B)(1 + \gamma_0^{-1})[1 + (1/2)(M_{\rm ism}/M_B)]}, \tag{2a}$$

where M_B and γ_0 are the initial values, respectively, of the mass and of the Lorentz gamma factor of the baryonic shell. Equations (1a)–(2a) differ from the ones derived by Blandford & McKee (1976) in a different framework but often quoted in the literature within the present context. Correspondingly, in the fully adiabatic case ($\epsilon = 0$), equations (1a)–(1d) have the following analytic solution (see Piran 1999 and Bianco & Ruffini 2005):

$$\gamma^2 = \frac{\gamma_0^2 + 2\gamma_0(M_{\rm ism}/M_B) + (M_{\rm ism}/M_B)^2}{1 + 2\gamma_0(M_{\rm ism}/M_B) + (M_{\rm ism}/M_B)^2}. \tag{2b}$$

In Bianco & Ruffini (2005) we have explicitly integrated the differential equations for $r(t)$ in equations (2a) and (2b), recalling that $\gamma^{-2} = 1 - [dr/(c\, dt)]^2$, where t is the time in the laboratory reference frame. We have then obtained new explicit analytic expressions for the equations of motion of the relativistic shell that are essential for explicitly obtaining the analytic expressions of the equitemporal surfaces in the fully radiative and fully adiabatic cases, respectively (see Bianco & Ruffini 2004, 2005).

3. APPROXIMATIONS ADOPTED IN THE CURRENT LITERATURE

We turn now to the comparison of the exact solutions given in equations (2a) and (2b) with the approximations used in the current literature. Following Blandford & McKee (1976), a so-called ultrarelativistic approximation $\gamma_0 \gg \gamma \gg 1$ has been widely adopted by many authors to solve equations (1a)–(1d) (see references in § 1). This leads to a simple constant-index power-law relation:

$$\gamma \propto r^{-a}, \tag{3}$$

with $a = 3$ in the fully radiative case and $a = 3/2$ in the fully adiabatic case. This simple relation is in contrast to the complexity of equations (2a) and (2b).

We now address the issue of establishing the domain of applicability of the simplified equation (3) used in the current literature in both the fully radiative case and the fully adiabatic case.

3.1. *The Fully Radiative Case*

We first consider the fully radiative case. If we assume

$$1/(\gamma_0 + 1) \ll M_{\rm ism}/M_B \ll \gamma_0/(\gamma_0 + 1) < 1, \tag{4}$$

in the numerator of equation (2a) the linear term in $M_{\rm ism}/M_B$ is negligible with respect to 1, and the quadratic term is *a fortiori* negligible, while in the denominator the linear term in $M_{\rm ism}/M_B$ is the leading one. Equation (2a) then becomes

$$\gamma \simeq [\gamma_0/(\gamma_0 + 1)]M_B/M_{\rm ism}. \tag{5}$$

If we multiply the terms of equation (4) by $(\gamma_0 + 1)/\gamma_0$, we obtain $1/\gamma_0 \ll (M_{\rm ism}/M_B)[(\gamma_0 + 1)/\gamma_0] \ll 1$, which is equivalent to $\gamma_0 \gg [\gamma_0/(\gamma_0 + 1)](M_B/M_{\rm ism}) \gg 1$ or, using equation (5), to

$$\gamma_0 \gg \gamma \gg 1, \tag{6}$$

which is indeed the inequality adopted in the "ultrarelativistic" approximation in the current literature. If we further assume $r^3 \gg r_0^3$, equation (5) can be further approximated by a simple constant-index power law as in equation (3):

$$\gamma \simeq [\gamma_0/(\gamma_0 + 1)]M_B/[(4/3)\pi n_{\rm ism} m_p r^3] \propto r^{-3}. \tag{7}$$

We turn now to the range of applicability of these approximations, consistent with the inequalities given in equation (4). It then becomes manifest that these inequalities can only be enforced in a finite range of $M_{\rm ism}/M_B$. The lower limit (LL) and the upper limit (UL) of such range can be conservatively estimated:

$$\left(\frac{M_{\rm ism}}{M_B}\right)_{\rm LL} = 10^2 \frac{1}{\gamma_0 + 1}, \quad \left(\frac{M_{\rm ism}}{M_B}\right)_{\rm UL} = 10^{-2} \frac{\gamma_0}{\gamma_0 + 1}. \tag{8a}$$

The allowed range of variability, if it exists, is then given by

$$\left(\frac{M_{\rm ism}}{M_B}\right)_{\rm UL} - \left(\frac{M_{\rm ism}}{M_B}\right)_{\rm LL} = 10^{-2} \frac{\gamma_0 - 10^4}{\gamma_0 + 1} > 0. \tag{8b}$$

A necessary condition for the applicability of the above approximations is therefore

$$\gamma_0 > 10^4. \tag{9}$$

It is important to emphasize that equation (9) is only a *necessary* condition for the applicability of the approximate equation (7), but it is not *sufficient*: equation (7) in fact can be applied only in a very limited range of r-values whose upper and lower limits are given in equation (8a). See for explicit examples § 4.

3.2. *The Adiabatic Case*

We now turn to the adiabatic case. If we assume

$$1/(2\gamma_0) \ll M_{\rm ism}/M_B \ll \gamma_0/2, \tag{10}$$

in the numerator of equation (2b) all terms are negligible with respect to γ_0^2, while in the denominator the leading term is the linear one in $M_{\rm ism}/M_B$. Equation (2b) then becomes

$$\gamma \simeq \sqrt{(\gamma_0/2)M_B/M_{\rm ism}}. \tag{11}$$

If we multiply the terms of equation (10) by $2/\gamma_0$, we obtain $1/\gamma_0^2 \ll (2/\gamma_0)(M_{\rm ism}/M_B) \ll 1$, which is equivalent to $\gamma_0^2 \gg (\gamma_0/2)(M_B/M_{\rm ism}) \gg 1$ or, using equation (11), to

$$\gamma_0^2 \gg \gamma^2 \gg 1. \tag{12}$$

If we now further assume $r^3 \gg r_0^3$, equation (11) can be further approximated by a simple constant-index power law as in equation (3):

$$\gamma \simeq \sqrt{(\gamma_0/2)M_B/[(4/3)\pi n_{\rm ism} m_p r^3]} \propto r^{-3/2}. \tag{13}$$

We turn now to the range of applicability of these approximations, consistent with the inequalities given in equation (10). It then becomes manifest that these inequalities can only be enforced in a finite range of $M_{\rm ism}/M_B$. The lower limit and the upper limit of such range can be conservatively estimated:

$$\left(\frac{M_{\rm ism}}{M_B}\right)_{\rm LL} = 10^2 \frac{1}{2\gamma_0}, \quad \left(\frac{M_{\rm ism}}{M_B}\right)_{\rm UL} = 10^{-2} \frac{\gamma_0}{2}. \tag{14a}$$

The allowed range of variability, if it exists, is then given by

$$\left(\frac{M_{\rm ism}}{M_B}\right)_{\rm UL} - \left(\frac{M_{\rm ism}}{M_B}\right)_{\rm LL} = 10^{-2} \frac{\gamma_0^2 - 10^4}{2\gamma_0} > 0. \tag{14b}$$

A necessary condition for the applicability of the above approximations is therefore

$$\gamma_0 > 10^2. \tag{15}$$

Again, it is important to emphasize that equation (15) is only a *necessary* condition for the applicability of the approximate equation (13), but it is not *sufficient*: equation (13) in fact can be applied only in a very limited range of r-values whose upper and lower limits are given in equation (14a). See for explicit examples § 4.

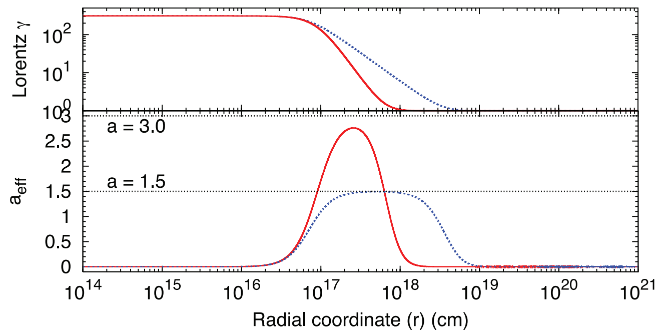

FIG. 1.—In the upper panel, the analytic behavior of the Lorentz γ factor during the afterglow era is plotted vs. the radial coordinate of the expanding thin baryonic shell in the fully radiative case of GRB 991216 (*solid red line*) and in the adiabatic case starting from the same initial conditions (*dotted blue line*). In the lower panel, the corresponding values of the effective power-law index $a_{\rm eff}$ (see eq. [16]), which is clearly not constant, is highly varying, and is systematically lower than the constant values of 3 and 3/2 purported in the current literature (*horizontal dotted black lines*), are plotted.

4. A SPECIFIC EXAMPLE

Having obtained the analytic expression of the Lorentz gamma factor for the fully radiative case in equation (2a), we illustrate in Figure 1 the corresponding gamma factor as a function of the radial coordinate in the afterglow era for GRB 991216 (see Ruffini et al. 2003 and references therein). We have also represented the corresponding solution that can be obtained in the adiabatic case, using equation (2b), starting from the same initial conditions. It is clear that, in both cases, there is not a simple power-law relation like equation (3) with a constant index a. We can at most define an "instantaneous" value $a_{\rm eff}$ for an "effective" power-law behavior:

$$a_{\rm eff} = -\frac{d\ln\gamma}{d\ln r}. \tag{16}$$

Such an effective power-law index of the exact solution smoothly varies from 0 to a maximum value that is always smaller than 3 or 3/2, in the fully radiative and adiabatic cases, respectively, and finally decreases back to 0 (see Fig. 1). In particular, from Figure 1, we see how in the fully radiative case the power-law index is consistently smaller than 3, and in the adiabatic case $a_{\rm eff} = 3/2$ is approached only for a small interval of the radial coordinate corresponding to the latest parts of the afterglow with a Lorentz gamma factor of the order of 10. In the case of GRB 991216, we have in fact $\gamma_0 = 310.13$, and neither equation (6) nor equation (12) can be satisfied for any value of r. Therefore, neither in the fully radiative case nor in the fully adiabatic case can the constant-index power-law expression in equation (3) be applied.

For clarity, we have integrated in Figure 2 an ideal GRB afterglow with the initial conditions as in GRB 991216 for selected higher values of the initial Lorentz gamma factor: $\gamma_0 = 10^3$, 10^5, 10^7, and 10^9. For $\gamma_0 = 10^3$, we then see that, again, in the fully radiative condition $a_{\rm eff} = 3$ is never reached and that in the adiabatic case $a_{\rm eff} \simeq 3/2$ is only reached in the region where $10 < \gamma < 50$. Similarly, for $\gamma_0 = 10^5$, in the fully radiative case $a_{\rm eff} \simeq 3$ is only reached around the point $\gamma = 10^2$, and in the adiabatic case $a_{\rm eff} \simeq 3/2$ for $10 < \gamma < 10^2$, although the non–power-law behavior still remains in the early and latest afterglow phases corresponding to the $\gamma \equiv \gamma_0$ and

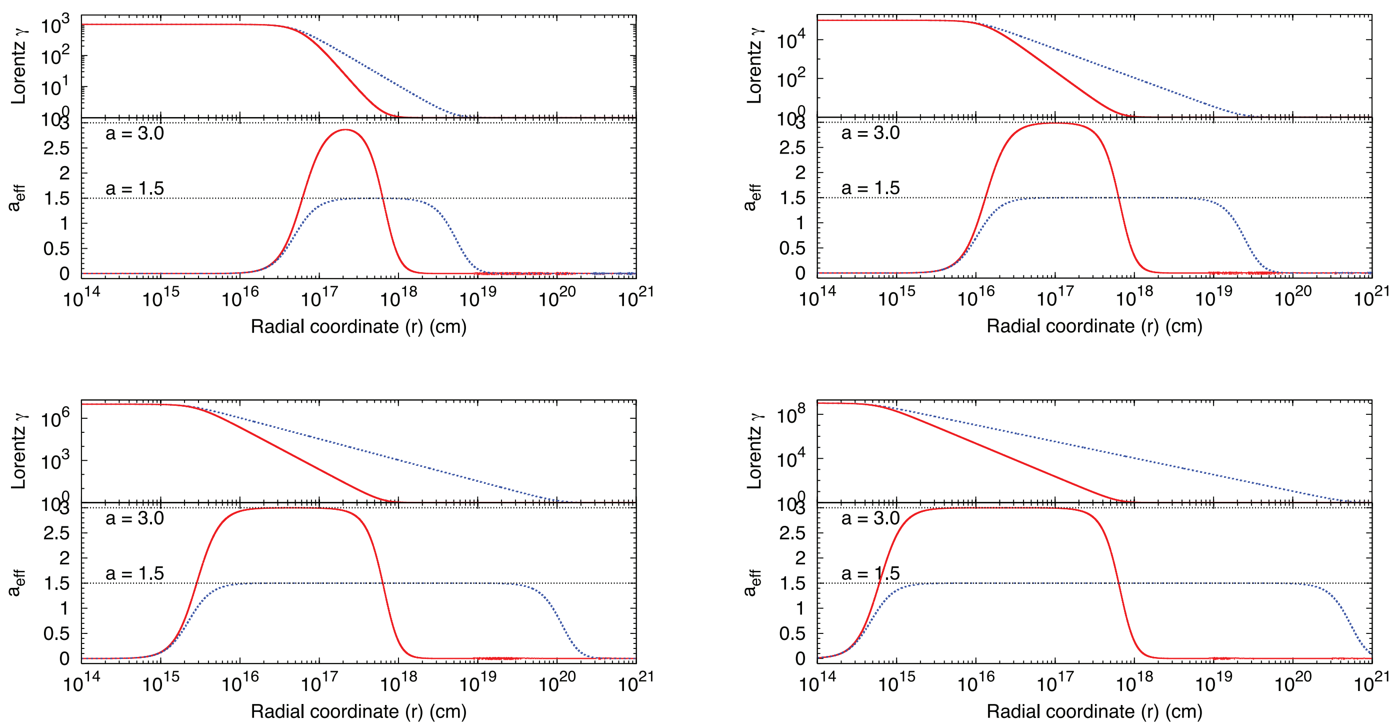

FIG. 2.—In these four diagrams we reproduce the same quantities plotted in Fig. 1 for four higher values of γ_0. The upper (lower) left diagram corresponds to $\gamma_0 = 10^3$ ($\gamma_0 = 10^5$). The upper (lower) right diagram corresponds to $\gamma_0 = 10^7$ ($\gamma_0 = 10^9$). It is manifest how asymptotically, by increasing the value of γ_0, the values $a = 3$ and $a = 3/2$ (*horizontal black dotted lines*) are reached, but only in a limited range of the radial coordinate and only for values of γ_0 much larger than the ones actually observed in GRBs.

$\gamma \to 1$ regimes, respectively. The same conclusion can be reached for the remaining cases $\gamma_0 = 10^7$ and $\gamma_0 = 10^9$.

We like to emphasize that the early part of the afterglow, where $\gamma \equiv \gamma_0$, which cannot be described by the constant-index power-law approximation, does indeed corresponds to the rising part of the afterglow bolometric luminosity and to its peak, which is reached as soon as the Lorentz gamma factor starts to decrease. We have shown (see, e.g., Ruffini et al. 2001, 2003, 2005b, and references therein) how the correct identification of the rising part of the afterglow and its peak is indeed crucial for the explanation of the observed "prompt radiation." Similarly, the power law cannot be applied during the entire approach to the Newtonian regime, which corresponds to some of the actual observations occurring in the latest afterglow phases.

5. CONCLUSIONS

It is well known that scaling laws and constant-index power-law expressions are obtainable only in the asymptotic case of ultrarelativistic regimes and in the Newtonian limit, while in the fully relativistic regime the scaling laws break down (see, e.g., Ruffini 1973). This circumstance is more subtle in GRB afterglows: (1) the ultrarelativistic approximation is only a necessary condition, but *not* a sufficient one, for the existence of scaling laws; (2) such a necessary condition implies values of the initial Lorentz gamma factor γ_0 outside the range currently observed in GRB sources.

We have shown in § 3.1 that, in the fully radiative case, the *necessary* ultrarelativistic condition for obtaining the appearance of scaling laws is $\gamma_0 > 10^4$. We recall that the γ_0-values deduced typically for GRBs are of the order of $\gamma_0 \simeq 10^2$ (see, e.g., GRB 030329, GRB 020322, GRB 991216, GRB 980519, GRB 980425, and GRB 970228; Ruffini et al. 2003, 2004, 2005a, and references therein). Thus, this necessary condition is never fulfilled in GRBs.

It would appear from § 3.2 that the *necessary* ultrarelativistic condition for obtaining the appearance of scaling laws is less severe in the adiabatic case: $\gamma_0 > 10^2$. However, this condition is not *sufficient* for the applicability of the constant-index power-law approximation to the entire afterglow, as clearly shown in Figure 1. The regime $a_{\rm eff} = 3/2$ is in fact approached only asymptotically and in a very limited region.

In the current literature (see references in § 1), a systematic use of the constant-index power-law approximation for the Lorentz gamma factor has been made. The $\gamma \equiv \gamma_0$ regime has been generally neglected or erroneously matched to the constant-index power-law approximation, hampering the understanding of the observed prompt radiation (see, e.g., Ruffini et al. 2001, 2003, 2005b).

We expect that the data from *Swift* will soon add observational evidence to the validity of this theoretical treatment.

We thank S. Blinnikov, J. Ehlers, and L. Titarchuk for discussions on the wording of our manuscript.

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